Numerical determination of the avalanche exponents of the Bak-Tang-Wiesenfeld model

S. Lübeck* and K. D. Usadel†
Theoretische Tieftemperaturphysik, Gerhard-Mercator-Universität Duisburg, Lotharstrasse 1, 47048 Duisburg, Germany

(Received 17 December 1996)

We consider the Bak-Tang-Wiesenfeld sandpile model on a two-dimensional square lattice of lattice sizes up to $L=4096$. A detailed analysis of the probability distribution of the size, area, duration, and radius of the avalanches will be given. To increase the accuracy of the determination of the avalanche exponents we introduce a new method for analyzing the data which reduces the finite-size effects of the measurements. The exponents of the avalanche distributions differ slightly from previous measurements and estimates obtained from a renormalization group approach. [S1063-651X(97)09604-9]

PACS number(s): 05.40.+j

I. INTRODUCTION

Bak, Tang, and Wiesenfeld (BTW) [1] introduced the concept of self-organized criticality (SOC) and realized it with the so-called ''sandpile model'' (BTW model). The steady state dynamics of the system is characterized by the probability distributions for the occurrence of relaxation clusters of a certain size, area, duration, etc. In the critical steady state these probability distributions exhibit power-law behavior. Using the concept of ''Abelian sandpile models'' [2] it is possible to calculate the static properties of the model exactly, e.g., the height probabilities, height correlations, number of steady state configurations, etc. [2–5]. However, the dynamical properties of the model, i.e., the exponents of the probability distributions, are not known exactly. Numerical simulations yield different values of the exponents depending on the considered system size and the used method of analyzing the data (see, for instance, [6–10]). Recently Pietronero *et al.* [11] introduced a renormalization scheme which allowed them to estimate the avalanche exponents. An improvement of this renormalization scheme was given by Ivashkevich [12] who obtained comparable results.

We investigate the original Bak-Tang-Wiesenfeld model on large lattice sizes ($L \leqslant 4096$) and measured the probability distributions. Since the numerical investigations of the BTW model by Manna [6] it is known that the obtained values of the exponents are affected by the finite size of the system. These finite-size effects have to be taken into account in order to get the ''real'' exponents. This has been done by extrapolation ($L \to \infty$) from data obtained for different L [6]. We could improve this method and are now able to measure the exponents of the infinite system directly, thus avoiding any extrapolation. In this way the accuracy of the obtained exponents is increased significantly. We also address the question whether the BTW model and the related sandpile models of Zhang [13] and Manna [14] belong to the same universality class. Finally, we discuss the assumption that the avalanche propagation can be described as a random walk.

II. MODEL

We consider the two-dimensional BTW model on a square lattice of size $L \times L$ in which integer variables $h_{i,j} \geqslant 0$ represent local heights. One perturbs the system by adding particles at a randomly chosen site $h_{i,j}$ according to

$$h_{i,j} \mapsto h_{i,j}+1 \quad \text{with random} \quad (i,j). \tag{1}$$

A site is called unstable if the corresponding height $h_{i,j}$ exceeds a critical value h_c, i.e., if $h_{i,j} \geqslant h_c$. Without loss of generality, we take $h_c=4$ throughout this work. An unstable site relaxes, its value is decreased by 4, and the neighboring sites are increased by one unit, i.e.,

$$h_{i,j} \to h_{i,j} - 4, \tag{2}$$

$$h_{i\pm 1,j\pm 1} \to h_{i\pm 1,j\pm 1} + 1, \tag{3}$$

where the update is done in parallel. We assume open boundary conditions with heights at the boundary fixed to zero.

System sizes from $L=64$ to $L=4096$ are investigated. Starting with a lattice of randomly distributed heights $h \in \{0,1,2,3\}$ the system is perturbed according to Eq. (1) and Dhar's ''burning algorithm'' is applied in order to check if the system has reached the critical steady state [2]. Then we start the actual measurements. All measurements are averaged over at least 10^6 nonzero avalanches except of the case $L=4096$ where only 5×10^5 measurements have been performed. We studied four different properties characterizing an avalanche. In the following we use the same notation as Majumdar and Dhar [7]. The total number of toppling events is called the size s of an avalanche. The number of distinct toppled lattice sites is denoted by s_d. Because a particular lattice site may topple several times, the number of toppling events exceeds the number of distinct toppled lattice sites, i.e., $s \geqslant s_d$. The duration t of an avalanche is equal to the number of update sweeps needed until all sites are stable again. The linear size of an avalanche r is measured via the radius of gyration of the avalanche cluster. In the critical steady state the corresponding probability distributions should obey power-law behavior characterized by exponents τ_s, τ_d, τ_t, and τ_r according to

$$P_s(s) \sim s^{-\tau_s}, \tag{4}$$

*Electronic address: sven@thp.uni-duisburg.de
†Electronic address: usadel@thp.uni-duisburg.de

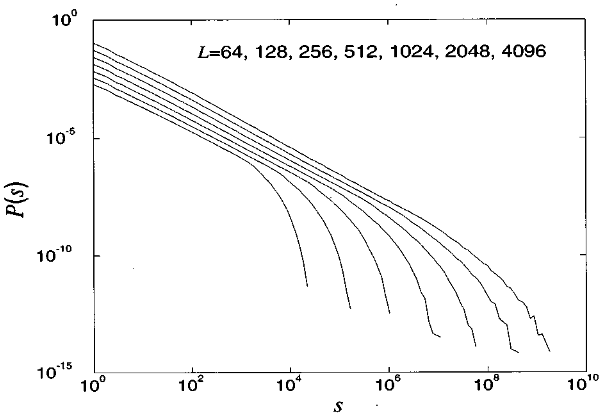

FIG. 1. The probability distribution $P(s)$ for different system sizes. The curves for $L<4096$ are shifted in the downward direction.

$$P_d(s_d) \sim s_d^{-\tau_d}, \tag{5}$$

$$P_t(t) \sim t^{-\tau_t}, \tag{6}$$

$$P_r(r) \sim r^{-\tau_r}. \tag{7}$$

III. SIMULATIONS AND RESULTS

Figure 1 displays the obtained results for the distribution $P_s(s)$ for different system sizes. A power-law fit to the straight portion of these curves yields the exponents $\tau_s(L)$. Figure 2 shows a plot of the exponents $\tau_s(L)$ vs $1/\ln L$. It is seen that for $L \geqslant 128$ the exponents obey the finite-size behavior

$$\tau_s(L) = \tau_{s,\infty} - \frac{\text{const}}{\ln L}, \tag{8}$$

as suggested already by Manna [6]. The extrapolation to $L \to \infty$ yields the value of the exponent $\tau_{s,\infty} = 1.247$. The probability distributions $P_d(s_d)$, $P_t(t)$, and $P_r(r)$ are analyzed in the same way with the result $\tau_{d,\infty} = 1.258$, $\tau_{t,\infty} = 1.405$, and $\tau_{r,\infty} = 1.588$, respectively. All exponents are

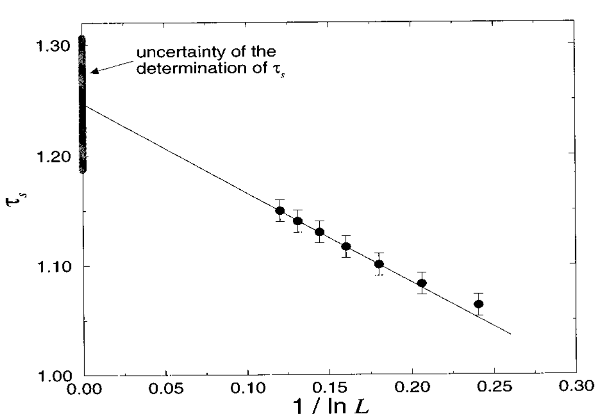

FIG. 2. Determination of the exponent τ using the extrapolation [Eq. (8)].

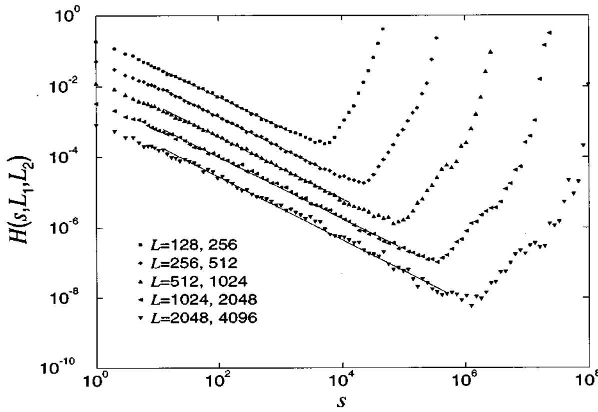

FIG. 3. The function $H(s,L_1,L_2)$ for different pairs L_1 and L_2. The curves are shifted with increasing system sizes in the downward direction. The solid lines correspond to a power-law fit. The obtained values of the exponent τ_s are listed in Table I.

slightly larger than those obtained from earlier simulations by Manna who considered smaller system sizes and had less statistics [6].

However, these values of the exponents are not very accurate. Namely, a crucial point in this analysis is the extension of the fit region in each distribution $P_s(s,L)$. Changing it, slightly different exponents are obtained. This uncertainty in the determination of the exponents $\tau(L)$ can be estimated to be at least of the order of ± 0.01. Taking then the propagation of these errors into account we can estimate the uncertainty in the determination of the extrapolated value τ_∞ to be of the order of ± 0.06 which is mainly due to the large distance of the measured values from the vertical axis (see Fig. 2). Thus it is in principle not possible to obtain the exponents of the BTW model with high accuracy by a simple extrapolation of the exponents via Eq. (8).

However, it is possible to improve the determination of the exponents not by using Eq. (8) for an extrapolation but for a direct determination of τ_∞. Consider for this purpose two probability distributions $P(s,L_1)$ and $P(s,L_2)$ corresponding to different system sizes with $L_1 > L_2$. If Eq. (8) describes the finite-size behavior of the exponents τ_s correctly, the probability distribution [Eq. (4)] for a given system size L behaves as

$$P(s,L) \sim s^{-\tau_{s,\infty}} s^{\text{const}/\ln L}. \tag{9}$$

Thus, the exponent $\tau_{s,\infty}$ can be determined directly by a power-law fit of the function $H(s,L_1,L_2)$ which is defined as

$$H(s,L_1,L_2) = \frac{P(s,L_1)^{\ln L_1}}{P(s,L_2)^{\ln L_2}} \sim s^{-\tau_{s,\infty}(\ln L_1 - \ln L_2)}. \tag{10}$$

In Fig. 3, $H(s,L_1,L_2)$ is plotted for various system sizes L_1 and L_2. A nice property of this function is that in contrast to the probability distribution the cutoff of the power-law behavior at large values of s is now very abrupt. We apply this analysis to all four distributions and the resulting exponents are listed in Table I. The values of the exponents $\tau_{s,\infty}$, $\tau_{t,\infty}$, and $\tau_{r,\infty}$ (except for the case $L_2 = 128$, $L_1 = 256$) fluctuate around their mean values given by $\tau_{s,\infty} = 1.293 \pm 0.009$, $\tau_{t,\infty} = 1.480 \pm 0.011$, and $\tau_{r,\infty} = 1.665$

TABLE I. Values of the exponents τ_s, τ_d, τ_t, and τ_r for different pairs of system sizes L.

L_1,L_2	$\tau_{s,\infty}$	$\tau_{d,\infty}$	$\tau_{t,\infty}$	$\tau_{r,\infty}$
128,256	1.293	1.253	1.486	1.183
256,512	1.281	1.287	1.464	1.665
512,1024	1.305	1.328	1.487	1.648
1024,2048	1.286	1.330	1.479	1.684
2048,4096	1.298	1.331	1.483	1.661

± 0.013. Only the exponent $\tau_{d,\infty}$ displays a significant L dependence. A possible origin of this L dependence is that Eq. (8) does not describe correctly the finite-size behavior of τ_d and that one has to add corrections to it. However, the data suggest that this additional L dependence vanishes for large system sizes and therefore the exponent saturates in the vicinity of $\tau_{d,\infty} \approx 1.33$. Note that the mentioned error bars describe only the statistical error. Because of the systematic errors, the real error bars are slightly larger.

IV. DISCUSSION

Despite their different toppling rules it is supposed that the BTW model, Zhang's model [13], and Manna's two-state model [14] belong to the same universality class; i.e., they should be characterized by the same exponents. Pietronero and co-workers [11] addressed this question by a renormalization group approach and found that the BTW model and Manna's two-state model belong to the same universality class. Different results were obtained by Ben-Hur and Biham [10] who found different values for the two models.

In Table II we compare our results with the exponents of the Zhang and the two-state model obtained from recent investigations on comparable lattice sites [15]. Within the error bars the BTW and Zhang models display the same exponents. The differences of the exponents τ_d and τ_r of the BTW and Manna's models cannot be explained by the error bars and thus we conclude that both models do not belong to the same universality class. But it is remarkable that both models display nearly the same duration exponent τ_t and especially that $\tau_t \approx \frac{3}{2}$. We assume that the value $\tau_t = \frac{3}{2}$ is a common feature of many sandpile models caused by an analogy of the avalanche propagation and a random walk, which we will discuss now.

The number of critical sites, $n(t)$, at a given update (time) step t can be considered as a random walker. Starting with $n(t=0)=1$ the avalanche performs a random walk $n(t=0) \rightarrow n(t=1) \rightarrow n(t=2) \rightarrow \cdots$ with the transition probabilities $p(n,n')$. The avalanche ceases to exist if the random walk returns to the origin $(n=0)$. In the simplest case the transition probabilities are homogeneous $p(n,n')=p(n-n')$, symmetric $p(\Delta n)=p(-\Delta n)$, and the random numbers Δn are uncorrelated. Then the avalanche probability distribution $P_t(t)$ is given by the probability $P_{\text{first return}}(t)$ that a random walker with initial value $n(t=0)=1$, with certain transition probabilities for increasing, decreasing, and maintaining n, returns for the first time to its starting point in t steps, which scales as [16]

$$P_{\text{first return}}(t) \sim t^{-3/2}. \tag{11}$$

Certain sandpile models are solved by an exact mapping of the avalanche propagation onto a simple random walk [17,18].

In Fig. 4 we present the number of critical sites vs update steps of a certain avalanche of the BTW model. The probability distribution $p(\Delta n)$ and the corresponding correlation function

$$C(\Delta t) = \frac{\langle \Delta n(t) \Delta n(t+\Delta t) \rangle}{\langle \Delta n^2 \rangle} \tag{12}$$

are shown in Fig. 5. The probability distribution $p(\Delta n)$ has to be symmetric in order to make sure that the random walk is recurrent; i.e., the probability that it ever returns to the origin is 1 [16]. The distribution displays asymmetries only for finite system sizes. A detailed analysis (not shown) yields that the third central moment of the distribution $p(\Delta n)$ tends to zero with diverging system size L, indicating that $p_{L\rightarrow\infty}(\Delta n)$ is symmetric.

The correlation function $C(\Delta t)$ is sharply peaked at $\Delta t=0$ but there are small oscillations for small values of Δt. Therefore, the second requirement for Eq. (11) to be valid, uncorrelated steps Δn, is only fulfilled approximately. This oscillating behavior is caused by the used parallel update process. Since toppling occurs at a given time step in one sublattice only, the update algorithm switches in sequential time steps between the two sublattices. The alternating correlation function indicates that the correlations within one

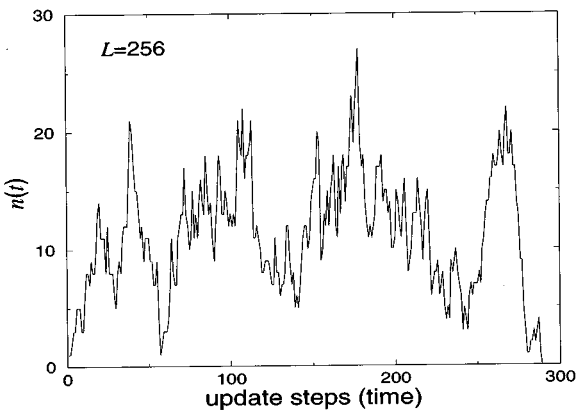

FIG. 4. The avalanche propagation as a random walk. The number of critical sites $n(t)$ is plotted against the update (time) steps for a certain avalanche of duration $t=289$. Starting from $n(t=0)=1$ the avalanche stops if the random walker returns to the origin for the first time.

TABLE II. Values of the exponents τ_s, τ_d, τ_t, and τ_r for the BTW model, Zhang's model, and Manna's two-state model. Because of the finite curvature of the probability distribution, the duration exponent τ_t of the Zhang model cannot be determined in the usual way [15].

Model	τ_s	τ_d	τ_t	τ_r
BTW	1.293	1.330	1.480	1.665
Zhang	1.282	1.338		1.682
Manna	1.275	1.373	1.493	1.743

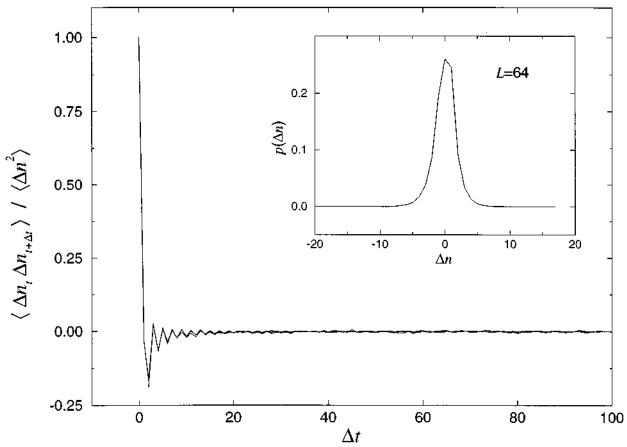

FIG. 5. The probability distribution $p(\Delta n)$ (see inset) and the correlations $C(\Delta t)$ for different system sizes ($L=64,128,256$).

sublattice differs from the correlation between the two sublattices. Thus, compared to the exactly solved sandpile models [17,18] where the correlation functions are simply given by a δ function the correlations of the BTW model are more complicated. But since these oscillations at small Δt have small amplitudes, we suggest that the avalanche propagation may be described as a random walk and that the exponent of the duration is $\tau_t=\frac{3}{2}$.

Scaling relations for the exponents τ_s, τ_d, τ_t, and τ_r can be obtained if one assumes that the size, area, duration, and radius scale as a power of each other, for instance,

$$t \sim r^{\gamma_{tr}}, \tag{13}$$

for the duration t of an avalanche and its radius r. The relation $P_t(t)dt=P_r(r)dr$ for the corresponding distribution functions leads to the scaling relation

$$\gamma_{tr}=\frac{\tau_r-1}{\tau_t-1}. \tag{14}$$

The exponents γ_{dr}, γ_{rs}, γ_{sd}, etc., are defined in the same way. The exponent γ_{tr} is usually identified with the dynamical exponent z and using a momentum-space analysis of the corresponding Langevin equations Díaz-Guilera showed that the dynamical exponent of the BTW and Zhang's models is given by $z=(d+2)/3$ [19]. On the other hand, one concludes from the compactness of the avalanche clusters that $\gamma_{dr}=2$. Thus one gets two scaling relations for the exponents τ_d, τ_r, and τ_t and using the result that $\tau_t=\frac{3}{2}$ the exponents of the probability distribution of the radius and the area are given by $\tau_r=\frac{5}{3}$ and $\tau_d=\frac{4}{3}$. These values are in good agreement with our numerical results and we would suggest that they are the exact exponents of the BTW model.

Majumdar and Dhar [7] assumed that the size and the area of an avalanche fulfill the relation

$$s \sim s_d n_c, \tag{15}$$

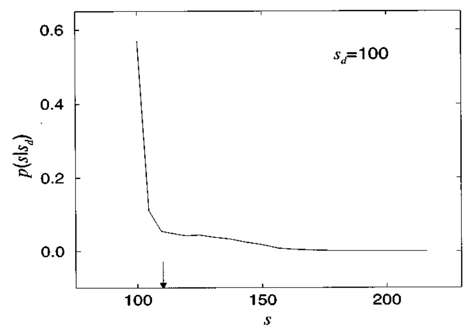

FIG. 6. The conditional probability distribution $p(s|s_d)$. The arrow marks the corresponding expectation value.

where n_c is the number of topplings at the site initiating the avalanche. If this equation holds, the exponents τ_s and τ_d have to fulfill the relation $\tau_s=2-1/\tau_d$. Using $\tau_d=\frac{4}{3}$ from above we obtain $\tau_s=\frac{5}{4}$ which is well outside the error bars of our numerical result, $\tau_s=1.293$. Thus we conclude that the assumed relation (15) does not describe the real scaling behavior.

Because of the lack of a scaling relation which connects τ_s with the other exponents, the exact value of the exponent τ_s is still unknown. Even a numerical determination of the exponent γ_{sd} yields useless results. The relation which defines γ_{sd} implies the assumption that the conditional probability distribution $p(s|s_d)$ is strongly peaked so that the expectation value $E(s|s_d)$ scales with the area s_d. Measurements of the conditional probabilities show that this is not the case for $p(s|s_d)$ (see Fig. 6). The distribution displays an asymmetric shape which violates the above assumptions.

A similar analysis of Manna's two-state model yielded that the dynamical exponent is given by $z\approx\frac{3}{2}$, resulting in $\tau_r=\frac{7}{4}$, $\tau_d=\frac{11}{8}$ [15]. The BTW model and the two-state model belong to different universality classes.

V. CONCLUSIONS

We studied numerically the dynamical properties of the BTW model on a two-dimensional square lattice and measured for large system sizes ($L\leqslant 4096$) the avalanche probability distributions. We introduced a new analysis to minimize the finite-size effects and determined the avalanche exponents with an improved accuracy. Our numerical results are consistent with the values $\tau_t=\frac{3}{2}$, $\tau_r=\frac{5}{3}$, and $\tau_d=\frac{4}{3}$ which we consider to be the exact exponents of the BTW model. We discussed the possibility that these values are caused by an analogy of the avalanche propagation and a random walk process. Further work has to be done to check this assumption. Recently, Ivashkevich [12] improved the renormalization group approach for sandpile models proposed by Pietronero *et al.* [11]. Both calculations yield the exponent $\tau_d\approx 1.25$, significantly smaller than our numerical estimates.

ACKNOWLEDGMENTS

This work was supported by the Deutsche Forschungsgemeinschaft through Sonderforschungsbereich 166, Germany. We would like to thank D. Ktitarev for helpful discussions.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).
[2] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[3] S. N. Majumdar and D. Dhar, J. Phys. A **24**, L357 (1991).
[4] V. B. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
[5] E. V. Ivashkevich, J. Phys. A **76**, 3643 (1994).
[6] S. S. Manna, Physica A **179**, 249 (1991).
[7] S. N. Majumdar and D. Dhar, Physica A **185**, 129 (1992).
[8] K. Christensen and Z. Olami, Phys. Rev. E **48**, 3361 (1993).
[9] V. B. Priezzhev, D. V. Ktitarev, and E. V. Ivashkevich, Phys. Rev. Lett. **76**, 2093 (1996).
[10] A. Ben-Hur and O. Biham, Phys. Rev. E **53**, R1317 (1996).
[11] L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994).
[12] E. V. Ivashkevich, Phys. Rev Lett. **76**, 3368 (1996).
[13] Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
[14] S. S. Manna, J. Phys. A **24**, L363 (1991).
[15] S. Lübeck (unpublished).
[16] G. R. Grimmet and D. R. Stirzaker, *Probability and Random Processes* (Clarendon Press, Oxford, 1992).
[17] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).
[18] H. Flyvbjerg, Phys. Rev. Lett. **76**, 940 (1996).
[19] A. Díaz-Guilera, Europhys. Lett. **26**, 177 (1994).